

Symmetries of the Energy-Momentum Tensor: Some Basic Facts

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Abstract

It has been pointed by Hall et al. [1] that matter collinations can be defined by using three different methods. But there arises the question of whether one studies matter collinations by using the $\mathcal{L}_\xi T_{ab} = 0$, or $\mathcal{L}_\xi T^{ab} = 0$ or $\mathcal{L}_\xi T_a^b = 0$. These alternative conditions are, of course, not generally equivalent. This problem has been explored by applying these three definitions to general static spherically symmetric spacetimes. We compare the results with each definition.

Keywords: Symmetries, Energy-Momentum Tensor

1 Introduction

It has been an interesting subject to use the symmetry group of a spacetime in constructing the solution of Einstein field equation (EFEs) given by

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}, \quad (1)$$

where G_{ab} are the components of the Einstein tensor, R_{ab} are the components of Ricci tensor and T_{ab} are the components of matter (energy-momentum) tensor, R is the Ricci scalar and κ is the gravitational constant. Further,

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these solutions are classified according to the Lie algebra structure generated by these symmetries. The well-known connection between Killing vectors (KVs) and constants of the motion [2,3] has encouraged the search for general relations between collineations and conservation laws [4]. Curvature and the Ricci tensors are the important quantities which play a significant role in understanding the geometric structure of spacetime. A pioneer study of curvature collineations (CCs) and Ricci collineations (RCs) has been carried out by Katzin, et al [5] and a further classification of CCs and RCs has been obtained by different authors [6,7].

The theoretical basis for the study of affine including Killing and homothetic vector fields is well understood and many examples are given. This is also true in the case of conformal fields as well as projective and curvature collineations. However, symmetries of the Ricci tensor and, in particular, energy-momentum tensor have recently been studied. In this paper we shall analyze the properties of a vector field along which the Lie derivative of the energy-momentum tensor vanishes, i.e., $\mathcal{L}_\xi T_{ab} = 0$. But a natural question arises whether one studies matter collineations defined in this way or those which satisfy $\mathcal{L}_\xi T^{ab} = 0$, or $\mathcal{L}_\xi T_a^b = 0$. These different definitions are not generally equivalent. We shall apply these three definitions to general static spherically symmetric spacetimes to check this observation. Since the energy-momentum tensor represents the matter part of the Einstein field equations and gives the matter field symmetries. Thus the study of matter collineations (MCs) seems more relevant from the physical point of view.

There is a growing interest in the study of MCs [1,8-11 and references therein]. Carot, et al [8] have discussed MCs from the point of view of the Lie algebra of vector fields generating them and, in particular, he discussed spacetimes with a degenerate T_{ab} . Hall, et al [1], in the discussion of RC and MC, have argued that the symmetries of the energy-momentum tensor may also provide some extra understanding of the the subject which has not been provided by KVs, Ricci and CCs. The same author also raised the question how to define matter collineation. Keeping this point in mind we address the problem of calculating MCs for static spherically symmetric spacetimes using the three different definitions. It is hoped that this would provide a better understanding of MCs.

The distribution of the paper follows. In the next section, we discuss some general issues about MCs and write down the MC equations. In section three we calculate MCs by solving MC equations for static spherically symmetric spacetimes using three different conditions. Final section carries a discussion

of the results obtained.

2 Some Basic Facts

Let (M, g) be a spacetime, M being a Hausdorff, connected, four dimensional manifold, and g a Lorentz metric with signature $(+, -, -, -)$.

A vector ξ is called a MC if the Lie derivative of the energy-momentum tensor along that vector is zero. That is,

$$\mathcal{L}_\xi T = 0, \quad (2)$$

where T is the energy-momentum tensor and \mathcal{L}_ξ denotes the Lie derivative along ξ of the energy-momentum tensor T . This equation, in a torsion-free space in a coordinate basis, reduces to a partial differential equation,

$$T_{ab,c}\xi^c + T_{ac}\xi^c_{,b} + T_{bc}\xi^c_{,a} = 0, \quad a, b, c = 0, 1, 2, 3. \quad (3)$$

where $,$ denotes partial derivative with respect to the respective coordinate. We shall also consider those symmetries generated by vector fields ξ satisfying $\mathcal{L}_\xi T^{ab} = 0$ or $\mathcal{L}_\xi T^b_a = 0$. These are ten coupled partial differential equations for four unknown functions (ξ^a) which are functions of all space-time coordinates in the case of covariant and contravariant forms but sixteen for mixed form.

Collineations can be proper or improper. A collineation of a given type is said to be *proper* if it does not belong to any of the subtypes. When we solve MC equations, solutions representing proper collineations can be found. However, in order to be related to a particular conservation law, and its corresponding constants of the motion, the *properness* of the collineation type must be known.

We know that every KV is an MC, but the converse is not always true. As given by Carot et al. [8], if T_{ab} is non-degenerate, $\det(T_{ab}) \neq 0$, the Lie algebra of the MCs is finite dimensional. If T_{ab} is degenerate, i.e., $\det(T_{ab}) = 0$, we cannot guarantee the finite dimensionality of the MCs. The study of MCs has many difficulties which can be listed as follows [1,12].

1. When we define affine and conformal vector fields on M , if the vector field is at least C^2 and C^3 respectively, then ξ is necessarily smooth on M . However, for any $k \in \mathbb{Z}^+$ there exists MC on spacetimes which are C^k not C^{k+1} . The same is true for the Ricci and CCs.

2. An affine and a conformal vector field ξ on M is uniquely determined by specifying it and its first covariant derivative and specifying it and its first and second covariant derivatives respectively at some $m \in M$. However, the value of ξ and its covariant derivatives of all orders at some $m \in M$ may not be enough to determine uniquely a MC ξ on M . Thus two MCs that agree on a non-empty open subset of M may not agree on M . These features are also found in RCs and CCS. This leads to a problem of the extendibility of local MCs to the whole of M which is more complicated than that for affine and conformal vector fields [7].
3. The set of all MCs on M is a vector space but, like the set of RCs or CCs and unlike the sets of affine and conformal vector fields, it may be infinite dimensional and may not be a Lie algebra. This latter defect arises from the fact that such collineations must be C^1 in order that their definitions make sense. But MCs (RCs and CCS) may be exactly C^1 and differentiability may be destroyed under the Lie bracket operation. If MCs are C^∞ then one recovers the Lie algebra structure but loses the non-smooth. The infinite dimensionality may also lead to problems related to the orbits of the resulting local diffeomorphism [7].
4. If the energy-momentum tensor is of rank 4, it may be regarded as a metric on M . Then the family of C^2 MCs is, in fact, a Lie algebra of smooth vector fields on M of finite dimension ≤ 10 and $\neq 9$.

It is obvious from the EFEs (1) that left hand side is the geometrical part constructed from the metric and its derivatives while the right hand side is the physical part describing the sources of the gravitational field. It is not clear whether (1) is to be written with the indices in the covariant, the contravariant or the mixed positions in any case. These lead to significant difficulties even with the definition of a matter symmetry. It has been shown that for almost all spacetimes (in a well defined topologically generic sense) the weyl tensor C and the energy-momentum tensor T (or the Einstein tensor G) determined the metric g uniquely up to a constant conformal factor and hence determined the Levi-Civita connection [13,14]. The special case of this result in vacuo is just Brinkmann's theorem [13,15]. The following theorem can be considered as an important result by considering the local diffeomorphisms associated with a vector field ξ on M [1,13,14].

Theorem: Let M be a spacetime manifold. Then, generically, any vector field ξ on M which simultaneously satisfies $\mathcal{L}_\xi T = 0$ ($\Leftrightarrow \mathcal{L}_\xi G = 0$) and

$\mathcal{L}_\xi C = 0$ is a homothetic vector field.

Thus with this concept of symmetry for all the gravitational sources, a metric symmetry (upto a constant homothetic scaling) generically results.

The most general spherically symmetric metric is given as [15]

$$ds^2 = e^{\nu(t,r)} dt^2 - e^{\lambda(t,r)} dr^2 - e^{\mu(t,r)} d\Omega^2, \quad (4)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. Since we are dealing with static spherically symmetric spacetimes, Eq.(4) reduces to

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - e^{\mu(r)} d\Omega^2. \quad (5)$$

3 Solution of MC Equations Using Three Definitions

In this section we shall use three different definitions to calculate MCs of static spherically symmetric spacetimes.

3.1 Solution When $\mathcal{L}_\xi T_{ab} = 0$

We can write MC Eqs.(3) in the expanded form as follows

$$T_{0,1}\xi^1 + 2T_0\xi_{,0}^0 = 0, \quad (6)$$

$$T_0\xi_{,1}^0 + T_1\xi_{,0}^1 = 0, \quad (7)$$

$$T_0\xi_{,2}^0 + T_2\xi_{,0}^2 = 0, \quad (8)$$

$$T_0\xi_{,3}^0 + \sin^2 \theta T_2\xi_{,0}^3 = 0, \quad (9)$$

$$T_{1,1}\xi^1 + 2T_1\xi_{,1}^1 = 0, \quad (10)$$

$$T_1\xi_{,2}^1 + T_2\xi_{,1}^2 = 0, \quad (11)$$

$$T_1\xi_{,3}^1 + \sin^2 \theta T_2\xi_{,1}^3 = 0, \quad (12)$$

$$T_{2,1}\xi^1 + 2T_2\xi_{,2}^2 = 0, \quad (13)$$

$$\xi_{,3}^2 + \sin^2 \theta \xi_{,2}^3 = 0, \quad (14)$$

$$T_{2,1}\xi^1 + 2T_2 \cot \theta \xi^2 + 2T_2\xi_{,3}^3 = 0, \quad (15)$$

where $T_3 = \sin^2 \theta T_2$. It is to be noticed that we are using the notation $T_{aa} = T_a$ etc. We solve these equations for the degenerate as well as the non-degenerate case. The nature of the solution of these equations changes when one (or more) T_a is zero. The nature changes even if $T_a \neq 0$ but $T_{a,1} = 0$.

A complete solution of MC Eqs.(6)-(15) has already been obtained [11,17] using this covariant definition both for the degenerate as well as non-degenerate cases. The degenerate case implies that $\det(T_{ab}) = 0$. When at least one of $T_a = 0$, we can have the following three main cases:

- (1) when only one of the $T_a \neq 0$,
- (2) when exactly two of the $T_a \neq 0$,
- (3) when exactly three of the $T_a \neq 0$.

It is mentioned here that the trivial case, where $T_a = 0$, shows that every vector field is an MC. For the sake of comparison, we would only give results in the form of tables at the end skipping all the details as these can be seen elsewhere [11,17].

3.2 Solution When $\mathcal{L}_\xi T^{ab} = 0$

In this case, MC equations can be written in the expanded form as follows:

$$T_{,1}^0 \xi^1 - 2T^0 \xi_{,0}^0 = 0, \quad (16)$$

$$T^0 \xi_{,0}^1 + T^1 \xi_{,1}^0 = 0, \quad (17)$$

$$T^0 \xi_{,0}^2 + T^2 \xi_{,2}^0 = 0, \quad (18)$$

$$T^0 \sin^2 \theta \xi_{,0}^3 + T^2 \xi_{,3}^0 = 0, \quad (19)$$

$$T_{,1}^1 \xi^1 - 2T^1 \xi_{,1}^1 = 0, \quad (20)$$

$$T^1 \xi_{,1}^2 + T^2 \xi_{,2}^1 = 0, \quad (21)$$

$$T^1 \sin^2 \theta \xi_{,1}^3 + T^2 \xi_{,3}^1 = 0, \quad (22)$$

$$T_{,1}^2 \xi^1 - 2T^2 \xi_{,2}^2 = 0, \quad (23)$$

$$T^2 (\sin^2 \theta \xi_{,2}^3 + \xi_{,3}^2) = 0, \quad (24)$$

$$\frac{T_{,1}^2}{2T^2} \xi^1 - \cot \theta \xi^2 - \xi_{,3}^3 = 0. \quad (25)$$

We solve this system of MC equations for the degenerate and non-degenerate cases skipping the algebra.

3.2.1 Degenerate Case

The degenerate case implies the following three main cases:

- (1) when only one of the $T^a \neq 0$,
- (2) when exactly two of the $T^a \neq 0$,
- (3) when exactly three of the $T^a \neq 0$.

Case 1: The first case gives either $T^0 \neq 0$, $T^i = 0$ ($i = 1, 2, 3$) or $T^1 \neq 0$, $T^j = 0$ ($j = 0, 2, 3$). When we use MC equations for the case (1a), we further have two possibilities, i.e., either $T^0 = \text{constant}$ or $T^0 \neq \text{constant}$. The first possibility yields

$$\xi^a = \xi^a(r, \theta, \phi). \quad (26)$$

The second option gives the following result

$$\xi^0 = (\ln \sqrt{T^0})' B(r, \theta, \phi) t + A(r, \theta, \phi), \quad \xi^i = \xi^i(r, \theta, \phi). \quad (27)$$

The case (1b), i.e., $T^1 \neq 0$, $T^j = 0$ gives the following two options according to $T^1 = \text{constant}$ or $T^1 \neq \text{constant}$. For $T^1 = \text{constant}$, we obtain ξ^a arbitrary. The case $T^1 \neq \text{constant}$ yields

$$\xi^1 = \sqrt{T^1} B(t, \theta, \phi), \quad \xi^j = \xi^j(t, \theta, \phi). \quad (28)$$

Case 2: This case has the following two possibilities:

(2a) $T^k = 0$, $T^l \neq 0$, (2b) $T^k \neq 0$, $T^l = 0$, ($k = 0, 1$)($l = 2, 3$) The first possibility gives either $T^2 = \text{constant} \neq 0$ or $T^2 \neq \text{constant}$. In the first case we obtain

$$\begin{aligned} \xi^k &= C(t, r), \\ \xi^2 &= A(t, r) \cos \phi + B(t, r) \sin \phi, \\ \xi^3 &= \cot \theta [-A(t, r) \sin \phi + B(t, r) \cos \phi] + D(t, r). \end{aligned} \quad (29)$$

In the second case, we have

$$\begin{aligned} \xi^0 &= C(t, r), \\ \xi^1 &= 0, \\ \xi^2 &= A(t, r) \cos \phi + B(t, r) \sin \phi, \\ \xi^3 &= \cot \theta [-A(t, r) \sin \phi + B(t, r) \cos \phi] + D(t, r). \end{aligned} \quad (30)$$

The case (2b) yields the following options:

- (i) $T^0 = \text{constant} \neq 0$, $T^1 = \text{constant} \neq 0$,
- (ii) $T^0 = \text{constant} \neq 0$, $T^1 \neq \text{constant}$,
- (iii) $T^0 \neq \text{constant}$, $T^1 = \text{constant} \neq 0$,
- (iv) $T^0 \neq \text{constant}$, $T^1 \neq \text{constant}$.

For (2bi), we have

$$\xi^a = \xi^a(\theta, \phi). \quad (31)$$

In the case (2bii), it follows that

$$\xi^j = \xi^j(\theta, \phi), \quad \xi^1 = \sqrt{T^1} A(\theta, \phi). \quad (32)$$

The case (2biii) further yields three options according to the value of α given by

$$\alpha = \frac{T^1}{T^0} \left(\frac{T^0}{2T^0} \right)' \quad (33)$$

either $\alpha < 0$, or $\alpha = 0$, or $\alpha > 0$. When $\alpha < 0$, we obtain the following solution

$$\begin{aligned} \xi^0 &= \frac{(\ln \sqrt{T^0})'}{\sqrt{\alpha}} [A(\theta, \phi) e^{\sqrt{\alpha} t} - B(\theta, \phi) e^{-\sqrt{\alpha} t}] + C(\theta, \phi), \\ \xi^1 &= [A(\theta, \phi) e^{\sqrt{\alpha} t} + B(\theta, \phi) e^{-\sqrt{\alpha} t}], \\ \xi^l &= \xi^l(\theta, \phi). \end{aligned} \quad (34)$$

For $\alpha = 0$, the solution is given by

$$\begin{aligned} \xi^0 &= (\ln \sqrt{T^0})' [A(\theta, \phi) \frac{t^2}{2} + B(\theta, \phi) t] + C(\theta, \phi), \\ \xi^1 &= A(\theta, \phi) t + B(\theta, \phi), \\ \xi^l &= \xi^l(\theta, \phi). \end{aligned} \quad (35)$$

When $\alpha > 0$, we have

$$\begin{aligned} \xi^0 &= \frac{(\ln \sqrt{T^0})'}{\sqrt{\alpha}} [A(\theta, \phi) \sin \sqrt{\alpha} t - B(\theta, \phi) \cos \sqrt{\alpha} t] + C(\theta, \phi), \\ \xi^1 &= [A(\theta, \phi) \cos \sqrt{\alpha} t + B(\theta, \phi) \sin \sqrt{\alpha} t], \\ \xi^l &= \xi^l(\theta, \phi). \end{aligned} \quad (36)$$

The last case, (2biv) when $T^{0'} \neq 0$, $T^{1'} \neq 0$, further yields three possibilities depending upon the value of β given by

$$\beta = \frac{\sqrt{T^1}[(\ln \sqrt{T^0})' \sqrt{T^1}]'}{T^0}, \quad (37)$$

i.e., either $\beta < 0$, or $\beta = 0$, or $\beta > 0$. The first possibility implies that

$$\begin{aligned} \xi^0 &= \frac{(\ln \sqrt{T^0})' \sqrt{T^1}}{\sqrt{\beta}} [A_1(\theta, \phi) e^{\sqrt{\beta}t} - A_2(\theta, \phi) e^{-\sqrt{\beta}t}] + A_3(\theta, \phi), \\ \xi^1 &= \sqrt{T^1} [A_1(\theta, \phi) e^{\sqrt{\beta}t} + A_2(\theta, \phi) e^{-\sqrt{\beta}t}], \\ \xi^l &= \xi^l(\theta, \phi). \end{aligned} \quad (38)$$

The second option yields the following solution

$$\begin{aligned} \xi^0 &= (\ln \sqrt{T^0})' \sqrt{T^1} [A_1(\theta, \phi) \frac{t^2}{2} + A_2(\theta, \phi) t] + A_3(\theta, \phi), \\ \xi^1 &= \sqrt{T^1} [A_1(\theta, \phi) t + A_2(\theta, \phi)], \\ \xi^l &= \xi^l(\theta, \phi). \end{aligned} \quad (39)$$

When $\beta > 0$, we obtain

$$\begin{aligned} \xi^0 &= \frac{(\ln \sqrt{T^0})' \sqrt{T^1}}{\sqrt{\beta}} [A_1(\theta, \phi) \sin \sqrt{\beta}t - A_2(\theta, \phi) \cos \sqrt{\beta}t] + A_3(\theta, \phi), \\ \xi^1 &= \sqrt{T^1} [A_1(\theta, \phi) \cos \sqrt{\beta}t + A_2(\theta, \phi) \sin \sqrt{\beta}t], \\ \xi^l &= \xi^l(\theta, \phi). \end{aligned} \quad (40)$$

Case 3: In this case we have three of $T^a \neq 0$ which further yields two cases.

$$(3a) \quad T^0 = 0, \quad T^i \neq 0, \quad (3b) \quad T^1 = 0, \quad T^j \neq 0.$$

When we take the first case (3a), MC equations give the following four possibilities:

- (3ai) $T^1 = \text{constant}, \quad T^2 = \text{constant},$
- (3aii) $T^1 \neq \text{constant}, \quad T^2 = \text{constant},$
- (3aiii) $T^1 = \text{constant}, \quad T^2 \neq \text{constant},$
- (3aiv) $T^1 \neq \text{constant}, \quad T^2 \neq \text{constant}.$

For (3ai), we obtain the following solution

$$\begin{aligned}\xi^k &= B(t), \\ \xi^2 &= [A_1(t) \cos \phi + A_2(t) \sin \phi], \\ \xi^3 &= -\cot \theta [A_1(t) \sin \phi - A_2(t) \cos \phi] + A_3(t).\end{aligned}\tag{41}$$

The case (3aii) gives the following result

$$\begin{aligned}\xi^0 &= A_1(t), \quad \xi^1 = \sqrt{T^1} A_2(t), \\ \xi^2 &= A_3(t) \cos \phi + A_4(t) \sin \phi, \\ \xi^3 &= -\cot \theta [A_3(t) \sin \phi - A_4(t) \cos \phi] + A_5(t).\end{aligned}\tag{42}$$

The cases (3aiii) and (3aiv) yield the same solution given by

$$\begin{aligned}\xi^0 &= A_1(t), \quad \xi^1 = 0, \\ \xi^2 &= A_2(t) \cos \phi + A_3(t) \sin \phi, \\ \xi^3 &= -\cot \theta [A_2(t) \sin \phi - A_3(t) \cos \phi] + A_4(t).\end{aligned}\tag{43}$$

The case (3b) also yields four possibilities given by

$$\begin{aligned}(3bi) \quad &T^0 = \text{constant}, \quad T^2 = \text{constant}, \\ (3bii) \quad &T^0 \neq \text{constant}, \quad T^2 = \text{constant}, \\ (3biii) \quad &T^0 = \text{constant}, \quad T^2 \neq \text{constant}, \\ (3biv) \quad &T^0 \neq \text{constant}, \quad T^2 \neq \text{constant}.\end{aligned}$$

The case (3bi) gives the following solution

$$\begin{aligned}\xi^0 &= A_1(r), \quad \xi^1 = A_2(r), \\ \xi^2 &= [A_3(r) \cos \phi + A_4(r) \sin \phi], \\ \xi^3 &= -\cot \theta [A_3(r) \sin \phi - A_4(r) \cos \phi] + A_5(r).\end{aligned}\tag{44}$$

For the case(3bii), it implies that

$$\begin{aligned}\xi^0 &= (\ln \sqrt{T^0})' A_1(r) t + A_2(r), \quad \xi^1 = A_1(r), \\ \xi^2 &= [A_3(r) \cos \phi + A_4(r) \sin \phi], \\ \xi^3 &= -\cot \theta [A_3(r) \sin \phi - A_4(r) \cos \phi] + A_5(r).\end{aligned}\tag{45}$$

The third and fourth possibilities give the same result

$$\begin{aligned}\xi^0 &= A_1(r), \quad \xi^1 = 0, \\ \xi^2 &= [A_2(r) \cos \phi + A_3(r) \sin \phi], \\ \xi^3 &= -\cot \theta [A_2(r) \sin \phi - A_3(r) \cos \phi] + A_4(r).\end{aligned}\tag{46}$$

3.2.2 Non-Degenrate Case

Solving Eqs.(16)-(20), we have

$$\xi^1 = \sqrt{T^1} A(t, \theta, \phi) \quad (47)$$

with

$$\ddot{A}(t, \theta, \phi) = -\alpha A(t, \theta, \phi), \quad \alpha = \frac{\sqrt{T^1}}{T^0} \left(\frac{T^{0'} \sqrt{T^1}}{2T^0} \right)'. \quad (48)$$

There arises three possibilities:

$$1. \quad \alpha < 0, \quad 2. \quad \alpha = 0, \quad 3. \quad \alpha > 0. \quad (49)$$

Case 1: In this case we take $\alpha = -\alpha$ so that we have

$$A(t, \theta, \phi) = A_1(\theta, \phi) \cosh \sqrt{\alpha} t + A_2(\theta, \phi) \sinh \sqrt{\alpha} t. \quad (50)$$

Using this value of A in the remaining MC equations, we get the following two cases, i.e. either $\frac{T^2}{T^0} = \text{constant}$ or $\frac{T^2}{T^0} \neq \text{constant}$.

In the first case (1a), MCs are given by

$$\begin{aligned} \xi^0 &= c_0, & \xi^1 &= 0, & \xi^2 &= c_1 \cos \phi + c_2 \sin \phi, \\ \xi^3 &= \cot \theta (-c_1 \sin \phi + c_2 \cos \phi) + c_3. \end{aligned} \quad (51)$$

Using the case (1b) together with MC equations we obtain the following two cases

$$\begin{aligned} (1bi) \quad & A_{1,2}(\theta, \phi) = 0 = A_{2,2}(\theta, \phi), \\ (1bii) \quad & \left(\frac{-2\alpha B + \beta}{2\alpha} \right) \left(\frac{T^2}{T^1} \right)' + \frac{T^2}{\sqrt{T^1}} = 0, \quad B = \int \frac{T^0}{\sqrt{T^1}} dr, \end{aligned}$$

where β is an integration constant. For (1bi), we have the following solution

$$\begin{aligned} \xi^0 &= \frac{T^{0'} \sqrt{T^1}}{2T^0 \sqrt{\alpha}} (c_4 \sinh \sqrt{\alpha} t + c_5 \cosh \sqrt{\alpha} t) + c_0, \\ \xi^1 &= \sqrt{T^1} (c_4 \cosh \sqrt{\alpha} t + c_5 \sinh \sqrt{\alpha} t), \\ \xi^2 &= c_1 \cos \phi + c_2 \sin \phi, \\ \xi^3 &= \cot \theta (-c_1 \sin \phi + c_2 \cos \phi) + c_3 \end{aligned} \quad (52)$$

with $T^2 = \text{constant}$. For $T^2 \neq \text{constant}$, it reduces to (1a). The case (1bii) gives

$$\eta = \frac{T^{2'} \sqrt{T^1} \alpha}{T^2 \epsilon (-2\alpha B + \beta)^2}, \quad (53)$$

where

$$\epsilon = \frac{T^2}{T^0 (-2\alpha B + \beta)}.$$

Eq.(53) implies that either (1bii*) $\eta > 0$, where η can be 1 or not 1, or (1bii**) $\eta < 0$. In the case (1bii*+) MCs are

$$\begin{aligned} \xi^0 &= \frac{(-2\alpha B + \beta)}{2\sqrt{\alpha}} [\{c_4 \cos \theta + (c_6 \cos \phi + c_7 \sin \phi) \sin \theta\} \sinh \sqrt{\alpha} t \\ &\quad + \{c_5 \cos \theta + (c_8 \cos \phi + c_9 \sin \phi) \sin \theta\} \cosh \sqrt{\alpha} t] + c_0, \\ \xi^1 &= \sqrt{T^1} [\{c_4 \cos \theta + (c_6 \cos \phi + c_7 \sin \phi) \sin \theta\} \cosh \sqrt{\alpha} t \\ &\quad + \{c_5 \cos \theta + (c_8 \cos \phi + c_9 \sin \phi) \sin \theta\} \sinh \sqrt{\alpha} t], \\ \xi^2 &= -\frac{\epsilon(-2\alpha B + \beta)^2}{2\alpha} [\{-c_4 \sin \theta + (c_6 \cos \phi + c_7 \sin \phi) \cos \theta\} \cosh \sqrt{\alpha} t \\ &\quad + \{-c_5 \sin \theta + (c_8 \cos \phi + c_9 \sin \phi) \cos \theta\} \sinh \sqrt{\alpha} t] + c_1 \cos \phi + c_2 \sin \phi, \\ \xi^3 &= \frac{-\epsilon(-2\alpha B + \beta)^2 \csc \theta}{2\alpha} [(-c_6 \sin \phi + c_7 \cos \phi) \cosh \sqrt{\alpha} t \\ &\quad + (-c_8 \sin \phi + c_9 \cos \phi) \sinh \sqrt{\alpha} t] + \cot \theta (-c_1 \sin \phi + c_2 \cos \phi) + c_3. \end{aligned} \quad (54)$$

The cases (1bii*++) and (1bii**) reduce to (1a).

Case 2: This case gives $\frac{T^{0'} \sqrt{T^1}}{2T^0} = \text{constant}$ which implies that either (2a) $\text{constant} = 0$ or (2b) $\text{constant} \neq 0$. The case (2a) yields $T^0 = \text{constant}$ which together with MC equations implies that either (2ai) $T^2 = \text{constant}$ or (2aii) $T^2 \neq \text{constant}$. It follows from (2ai) that either $T^1 = \text{constant}$ or $T^1 \neq \text{constant}$. The case (2ai*) gives the following MCs

$$\begin{aligned} \xi^0 &= -ac_4 \frac{r}{\sqrt{b}} + c_0, \quad \xi^1 = \sqrt{b}(c_4 t + c_5), \\ \xi^2 &= c_1 \cos \phi + c_2 \sin \phi, \quad \xi^3 = \cot \theta (-c_1 \sin \phi + c_2 \cos \phi) + c_3. \end{aligned} \quad (55)$$

For (2ai**), we obtain the following solution

$$\begin{aligned} \xi^0 &= -ac_4 \int \frac{dr}{\sqrt{T^1}} + c_0, \quad \xi^1 = \sqrt{T^1}(c_4 t + c_5), \\ \xi^2 &= c_1 \cos \phi + c_2 \sin \phi, \quad \xi^3 = \cot \theta (-c_1 \sin \phi + c_2 \cos \phi) + c_3. \end{aligned} \quad (56)$$

The case (2a_{ii}) yields the the same solution as for the case (1a).

The case (2b) further gives the following two possibilities

$$(T^2)' - \frac{2cT^2}{\sqrt{T^1}} = 0, \quad (T^2)' - \frac{2cT^2}{\sqrt{T^1}} \neq 0 \quad (57)$$

The case (2b_i) reduces to (1a). For (2b_{ii}), we obtain

$$\begin{aligned} \xi^0 &= \epsilon_1 \left(c_4 \frac{t^2}{2} + c_5 t \right) - \frac{c_4 \lambda_1 e^{2\epsilon_1 \int \frac{dr}{\sqrt{T^1}}}}{2\epsilon_1} + c_0, \quad \xi^1 = \sqrt{T^1} (c_4 t + c_5), \\ \xi^2 &= c_1 \cos \phi + c_2 \sin \phi, \quad \xi^3 = \cot \theta (-c_1 \sin \phi + c_2 \cos \phi) + c_3, \end{aligned} \quad (58)$$

where $\epsilon_1 = \frac{T^{0'} \sqrt{T^1}}{2T^0} = \text{constant}$.

Case 3: This case is very similar to the case 1 and can be solved on the same lines.

3.3 Solution When $\mathcal{L}_\xi T_b^a = 0$

For this definition, MC equations take the following expanded form

$$T^0_{0,1} \xi^1 = 0, \quad (59)$$

$$(T^0_0 - T^1_1) \xi^0_{,1} = 0, \quad (60)$$

$$(T^0_0 - T^2_2) \xi^0_{,2} = 0, \quad (61)$$

$$(T^0_0 - T^3_3) \xi^0_{,3} = 0, \quad (62)$$

$$(T^1_1 - T^0_0) \xi^1_{,0} = 0, \quad (63)$$

$$T^1_{1,1} \xi^1 = 0, \quad (64)$$

$$(T^1_1 - T^2_2) \xi^1_{,2} = 0, \quad (65)$$

$$(T^1_1 - T^3_3) \xi^1_{,3} = 0, \quad (66)$$

$$(T^2_2 - T^0_0) \xi^2_{,0} = 0, \quad (67)$$

$$(T^2_2 - T^1_1) \xi^2_{,1} = 0, \quad (68)$$

$$T^2_{2,1} \xi^1 = 0, \quad (69)$$

$$(T^3_3 - T^0_0) \xi^3_{,0} = 0, \quad (70)$$

$$(T^3_3 - T^1_1) \xi^3_{,1} = 0, \quad (71)$$

$$T^3_{3,1} \xi^1 = 0. \quad (72)$$

When we solve these MC equations, after some algebra, we obtain arbitrary MCs for all the possibilities of degenerate and non-degenerate cases [18].

4 Conclusion

We know that the metric tensor is non-degenerate whereas the Ricci, Riemann and energy-momentum tensors are not necessarily non-degenerate. When there is a degeneracy, it is possible to have arbitrary collineations. Thus KVs will always be definite but collineations can be indefinite. Further, if the relevant tensor vanishes, all vectors become collineations as for Minkowski space where every vector is a MC. Also, for vacuum spacetime every vector will be an MC as it is Ricci flat (e.g. Schwarzschild metric).

There is a problem with respect to the definition of MC because of the possible choices T_{ab} , T^{ab} , T^a_b . In this paper we have evaluated MCs for static spherically symmetric spacetimes using the three definitions. The motivation behind this is two fold: First to check whether these three definitions give similar results. If not the same which one gives the more interesting results. This would help us to understand the distribution of matter symmetries and comparison to the symmetries of the metric, Ricci and curvature tensors. We discuss the results obtained using the table given below:

Table 1. MCs using $\mathcal{L}_\xi T_{ab} = 0$, $\mathcal{L}_\xi T^{ab} = 0$ or $\mathcal{L}_\xi T^a_b = 0$ for the Degenerate Case

| Cases | $\mathcal{L}_\xi T_{ab} = 0$ | $\mathcal{L}_\xi T^{ab} = 0$ | $\mathcal{L}_\xi T^a_b = 0$ |
|--------|------------------------------|------------------------------|-----------------------------|
| 1ai | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 1aai | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 1b | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 2ai | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 2aai | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 2bi | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 2bii | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 2biii | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 3ai | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 3aai* | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 3aai** | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 3aiii | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 3bi | 4 | Infinite-dimensionl | Infinite-dimensionl |
| 3bii | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 3biii | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |
| 3biv | Infinite-dimensionl | Infinite-dimensionl | Infinite-dimensionl |

Table 2. MCs using $\mathcal{L}_\xi T_{ab} = 0$, $\mathcal{L}_\xi T^{ab} = 0$ or $\mathcal{L}_\xi T_a^b = 0$ for the Non-degenerate Case

| Cases | $\mathcal{L}_\xi T_{ab} = 0$ | Cases | $\mathcal{L}_\xi T^{ab} = 0$ | $\mathcal{L}_\xi T_a^b = 0$ |
|--------|------------------------------|-----------|------------------------------|-----------------------------|
| 1a | 6 | 1a | 4 | Infinite-dimensionl |
| 1bi | 4 | 1bi | 6 | Infinite-dimensionl |
| 1bii* | 6 | 1bii*+ | 10 | Infinite-dimensionl |
| 1bii** | 6 | 1bii*++ | 4 | Infinite-dimensionl |
| 1biii | 4 | 1bii** | 4 | Infinite-dimensionl |
| 2ai | 4 | 2ai* | 6 | Infinite-dimensionl |
| 2aii | 4 | 2ai** | 6 | Infinite-dimensionl |
| 2b | 10 | 2aii, 2bi | 4 | Infinite-dimensionl |
| 2c | 4 | 2bii | 6 | Infinite-dimensionl |
| 3 | 4 | 3 | Same as 1 | Infinite-dimensionl |

From these tables we see that out of three definitions, the mixed form gives an arbitrary MC in all cases. The contravariant form yields infinite dimensional MCs for the degenerate case but finite for the non-degenerate case. For the covariant definition, there exists an interesting case which gives finite number of MCs in one possibility even for the degenerate case. The non-degenerate case gives finite number of MCs in covariant form. We know that if T_{ab} is non-degenerate, the Lie algebra of the MCs is finite dimensional but if T_{ab} is degenerate, finite dimensionality of MCs cannot be guaranteed. The covariant and contravariant definitions verify this statement but the mixed form yields infinite dimensional MCs even for the non-degenerate case. It is always an interesting feature if there exist finite MCs in the degenerate case and the covariant definition gives such a possibility. This feature obviously motivates one to use this definition for the classification of spacetimes according to MCs. It is worth mentioning here that the purely covariant and contravariant cases give exactly the same class of MCs for the non-degenerate case.

We would like to comment the degenerate case a little more. It has been shown that, in general, there are infinity many MCs which musty be found by the solution of relevant MC equations. However, the MCs in the degenerate case are not as useful as the ones of the non-degenerate case. Indeed the assumption of the degeneracy of T_{ab} leads to differential equations which fix the metric functions up to arbitrary constant of integration. Hence the form of the matter tensor can be determined making the constraint imposed

the MC redundant. It has been shown [9] that only interesting case for the degenerate case is when $\text{rank } T_{ab} = 1$, i.e., a null Einstein-Maxwell field or a dust fluid.

One may ask whether there can exist vectors which are simultaneously solutions of $\mathcal{L}_\xi T_{ab} = 0$ and $\mathcal{L}_\xi T^{ab} = 0$ or of $\mathcal{L}_\xi T_a^b = 0$. We see from table 1, for the degenerate case, the only common solutions of $\mathcal{L}_\xi T_{ab} = 0$ and $\mathcal{L}_\xi T^{ab} = 0$ are the arbitrary collineations and for the non-degenerate case (table 2) the only common solutions are necessarily isometries. For $\mathcal{L}_\xi T_{ab} = 0$ and $\mathcal{L}_\xi T_a^b = 0$, the only common solution are the arbitrary collineations (degenerate case) but there is no common solution for the non degenerate case.

The infinite dimensionality of MCs may lead to different problems, e.g., one cannot define Lie algebra. Also, this may lead to problems related to the orbits of the resulting local diffeomorphism [7, 12]. In the light of such problems and the usefulness of the covariant definition for MC [10,11] it can be concluded that the covariant definition of MC should be preferred.

Acknowledgment

One of the authors (MS) would like to thank Prof. Graham S. Hall for the useful discussion on the topic.

References

- [1] Hall, G.S., Roy, I. and Vaz, L.R.: Gen. Rel and Grav. **28**(1996)299.
- [2] Noether, E.: Nachr, Akad. Wiss. Gottingen, II, Math. Phys. **K12**(1918)235.
- [3] Davis, W.R. and Katzin, G.H.: Am. J. Phys. **30**(1962)750.
- [4] Bokhari, A.H. and Qadir, Asghar: J. Math. Phys. **34**(1993)3543; J. Math. Phys. **31**(1990)1463.
- [5] Katzin, G.H., Levine, J. and Davis, H.R.: J. Math. Phys. **10**(1969)617.
- [6] Bokhari, A.H., Kashif, A.R. and Qadir, Asghar: J. Math. Phys. **41**(2000)2167.
- [7] Hall, G.S. and da Costa, J.: J. Math. Phys. **32**(1991)2848;2854.

- [8] Carot, J. and da Costa, J.: *Procs. of the 6th Canadian Conf. on General Relativity and Relativistic Astrophysics*, Fields Inst. Commun. 15, Amer. Math. Soc. WC Providence, RI(1997)179.
- [9] Carot, J., da Costa, J. and Vaz, E.G.L.R.: *J. Math. Phys.* **35**(1994)4832.
- [10] Camci, U. and Sharif, M.: *Gen. Rel. and Grav.* **35**(2003)97;
 Camci, U. and Sharif, M.: *Class. Quantum Grav.* **20**(2003)2169;
 Sharif, M.: *J. Math. Phys.* **44**(2003)5141; *J. Math. Phys.* **45**(2004)1518;
J. Math. Phys. **45**(2004)1532.
- [11] Sharif, M. and Aziz, Sehar: *Gen. Rel. and Grav.* **35**(2003)1093.
- [12] Hall, G.S.: *Symmetries and Curvature Structure in General Relativity* (World Scientific, 2004).
- [13] Hall, G.S.: *Lecture at Conference on Differential Geometry* (University of Coimbra, Portugal, 1984).
- [14] Hall, G.S. and Rendall, A.D.: *J. Math. Phys.* **28**(1987)1837.
- [15] Ehlers, J. and Kundt, W.: *In Gravitation: An Introduction to Current Research*, ed. Witten, L. (Wiley, New York, 1962).
- [16] Stephani, H., Kramer, D., Maccallum, M., Hoenselaers, C. and Herlt, E. *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, 2003).
- [17] Aziz, Sehar: M.Phil. Thesis (University of the Punjab, 2002).
- [18] Ismaeel, Tariq: M.Phil. Thesis (University of the Punjab, 2006).